

# JACOBIANS AMONG ABELIAN THREEFOLDS: A GEOMETRIC APPROACH

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ABSTRACT. Let  $(A, \theta)$  be a principally polarized abelian threefold over a perfect field  $k$ , not isomorphic to a product over  $\bar{k}$ . There exists a canonical extension  $k'/k$ , of degree  $\leq 2$ , such that  $(A, \theta)$  becomes isomorphic to a Jacobian over  $k'$ . The aim of this note is to give a geometric construction of this extension.

## 1. INTRODUCTION

Let  $(A, \theta)$  be a principally polarized abelian variety of dimension 3 over a field  $k$ . If  $k$  is algebraically closed,  $(A, \theta)$  is the Jacobian variety of a curve  $C$  (or a product of Jacobians). If  $k$  is an arbitrary perfect field the situation is more subtle (see Proposition 3.1 below): there is still a curve  $C$  defined over  $k$ , but either  $(A, \theta)$  is isomorphic to  $JC$ , or they become isomorphic only after a quadratic extension  $k'$  of  $k$ , uniquely determined by  $(A, \theta)$ .

Now given  $(A, \theta)$ , how can we decide if it is a Jacobian, and more precisely determine the extension  $k'/k$ ? For  $k \subset \mathbb{C}$ , a solution is given in [LRZ] in terms of modular forms. Here we propose a geometric approach, based on a construction of Recillas. We have to assume that  $A$  admits a rational theta divisor  $\Theta$ , and a rational point  $a \in A(k)$  outside a certain explicit divisor  $\Sigma_A \subset A$ . This guarantees that the curve  $\Theta \cap (\Theta + a)$  is smooth, and that there exists  $b \in A(k)$  such that the involution  $z \mapsto b - z$  acts freely on that curve. The quotient  $X_a$  is a non-hyperelliptic genus 4 curve; its canonical model lies on a unique quadric  $Q \subset \mathbb{P}^3$ . Then for  $\text{char}(k) \neq 2$  the extension  $k'$  is  $k(\sqrt{\text{disc}(Q)})$  (we will give more detailed statements in §3).

The proof has two steps. We consider first the case where  $(A, \theta)$  is a Jacobian, and prove that in that case the quadric  $Q$  is  $k$ -isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  (§2). Then we treat the case where  $(A, \theta)$  is *not* a Jacobian, and prove that the nontrivial automorphism of the extension  $k'/k$  exchanges the two rulings of  $Q$  (§3); this is enough to prove the theorem.

## 2. RECILLAS' CONSTRUCTION

Throughout the paper we work over a perfect field  $k$ .

In this section we fix a non-hyperelliptic curve  $C$  of genus 3 (that is, a smooth plane quartic curve), defined over  $k$ . We will denote by  $K$  its canonical class. We assume that the principal polarization of  $JC$  can be defined by a theta divisor  $\Theta$  defined over  $k$ . There exists a degree 2 divisor class  $D$  such that  $\Theta$  is the image of  $\text{Sym}^2 C - D$  in  $JC$ ; this class is unique, hence defined over  $k$ . Note that since  $C$  is not hyperelliptic,  $\Theta$  is smooth and the map  $E \mapsto E - D$  induces an isomorphism of  $\text{Sym}^2 C$  onto  $\Theta$ .

We choose a point  $a \in JC(k)$ ; we are interested in the curve  $\tilde{X}_a := \Theta \cap (\Theta + a)$ . Put  $b = K + a - 2D \in JC(k)$ ; we have  $-\Theta = \Theta + a - b$ . The involution  $z \mapsto b - z$  exchanges  $\Theta$  and  $\Theta + a$ , hence induces an involution  $\iota$  of  $\tilde{X}_a$ . We define a divisor  $\Sigma_{JC} \subset JC$  by  $\Sigma_{JC} = \Sigma'_{JC} \cup \Sigma''_{JC}$ , where

$$\Sigma'_{JC} = \{2E - K \mid E \in \text{Sym}^2 C\} \quad \text{and} \quad \Sigma''_{JC} = C - C.$$

**Proposition 2.1.** *The curve  $\tilde{X}_a := \Theta \cap (\Theta + a)$  is smooth and connected if and only if  $a \in JC \setminus \Sigma_{JC}$ . If this is the case, the involution  $\iota$  of  $\tilde{X}_a$  is fixed point free.*

*Proof :* Throughout the paper it will be convenient to use the following notation: given a divisor class  $d$  of degree 2 on  $C$  with  $h^0(\mathcal{O}_C(d)) = 1$ , we denote by  $\langle d \rangle \in \text{Sym}^2 C$  the unique effective divisor in the class  $d$ .

Let  $z \in \tilde{X}_a$ . By [K], thm. 2, the tangent space  $\mathbb{P}T_z(\Theta) \subset \mathbb{P}T_z(JC) = \mathbb{P}^2$  is identified with the line spanned by the divisor  $\langle D + z \rangle \in \text{Sym}^2 C$ . Similarly  $\mathbb{P}T_z(\Theta + a)$  is identified with the line spanned by  $\langle D + z - a \rangle \in \text{Sym}^2 C$ ; the intersection  $\tilde{X}_a$  is singular at  $z$  if and only if these two lines coincide. If this happens, then either

- the two divisors  $\langle D + z \rangle$  and  $\langle D + z - a \rangle$  have a common point, which implies  $a \in C - C$ ; or

- $\langle D + z \rangle + \langle D + z - a \rangle \sim K$ , which implies  $a \in \Sigma'_{JC}$ .

Conversely, if  $a \in \Sigma'_{JC}$ , we have  $K + a \sim 2E$  with  $E \in \text{Sym}^2 C$ ; then  $z = E - D$  is a singular point of  $\tilde{X}_a$ . If  $a \sim p - q$ , with  $p, q \in C$ , the intersection  $\tilde{X}_a$  is reducible, equal to  $(C + p - D) \cup (K - D - q - C)$ .

Assume now  $a \notin \Sigma_{JC}$ , so that  $\tilde{X}_a$  is smooth; the exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{O}_{JC}(-\Theta - (\Theta + a)) \rightarrow \mathcal{O}_{JC}(-\Theta) \oplus \mathcal{O}_{JC}(-(\Theta + a)) \rightarrow \\ \rightarrow \mathcal{O}_{JC} \rightarrow \mathcal{O}_{\tilde{X}_a} \rightarrow 0 \end{aligned}$$

gives  $h^0(\mathcal{O}_{\tilde{X}_a}) = 1$ , hence  $\tilde{X}_a$  is connected. If a point  $z \in \tilde{X}_a$  is fixed by  $\iota$  it satisfies  $2(D + z) \sim K + a$ , which implies  $a \in \Sigma'_{JC}$ . ■

2.2. We assume from now on  $a \notin \Sigma_{JC}$ . We denote by  $X_a$  the quotient curve  $\tilde{X}_a/\iota$ . The adjunction formula gives

$$K_{\tilde{X}_a} \sim (\Theta + (\Theta + a))|_{\tilde{X}_a} = 2\Theta^3 = 12, \quad \text{hence } g(\tilde{X}_a) = 7 \text{ and } g(X_a) = 4.$$

If  $\text{char}(k) \neq 2$ , the principally polarized abelian variety  $JC$  is canonically isomorphic to the Prym variety associated to the double covering  $\tilde{X}_a \rightarrow X_a$  ([B], §3.b; [R]).

We embed  $\tilde{X}_a$  into  $\text{Sym}^2 C \times \text{Sym}^2 C$  by  $z \mapsto (\langle D + z \rangle, \langle K + a - (D + z) \rangle)$ . Then  $\tilde{X}_a$  is identified with  $s^{-1}(|K + a|)$ , where  $s : \text{Sym}^2 C \times \text{Sym}^2 C \rightarrow \text{Sym}^4 C$  is the sum map. The involution  $\iota$  is induced by the involution of  $\text{Sym}^2 C \times \text{Sym}^2 C$  which exchanges the factors. The map  $s : \tilde{X}_a \rightarrow |K + a|$  factors through  $\iota$ , hence induces a 3-to-1 map  $t : X_a \rightarrow |K + a|$  ( $\cong \mathbb{P}^1$ ). The fibre of  $t$  above  $E \in |K + a|$  parametrizes the decompositions  $E = d + d'$ , with  $d, d'$  in  $\text{Sym}^2 C$ .

We now consider the involution  $(d, d') \mapsto (\langle K - d \rangle, \langle K - d' \rangle)$  of  $\text{Sym}^2 C \times \text{Sym}^2 C$ ; it maps  $\tilde{X}_a$  onto  $s^{-1}(|K - a|) = \tilde{X}_{-a}$  and commutes with  $\iota$ , hence induces an isomorphism  $X_a \xrightarrow{\sim} X_{-a}$ . By composition with the map  $X_{-a} \rightarrow |K - a|$  defined above we obtain another degree 3 map  $t' : X_a \rightarrow |K - a|$ .

The maps  $t$  and  $t'$  are defined over  $k$ ; they define two  $g_3^1$  on  $X_a$ , that is, two linear series of degree 3 and projective dimension 1, defined over  $k$ .

**Lemma 2.3.** *The two  $g_3^1$  defined by  $t$  and  $t'$  on  $X_a$  are distinct.*

*Proof :* Let us first observe that the degree 4 morphism  $f : C \rightarrow \mathbb{P}^1$  defined by the linear system  $|K + a|$  is separable. If this is not the case, we have  $\text{char}(k) = 2$  and  $f$  factors as  $C \xrightarrow{F} C_1 \xrightarrow{g} \mathbb{P}^1$ , where  $C_1/k$  is the pull back of  $C/k$  by the automorphism  $\lambda \mapsto \lambda^2$  of  $k$ ,  $F$  is the Frobenius  $k$ -morphism and  $g$  is separable of degree 2 (see [H], IV.2). But then  $C_1$  is hyperelliptic, hence also  $C$ .

Assume that the two linear series are the same. By the previous observation there exists a divisor  $E = p + q + r + s$  in  $|K + a|$  consisting of 4 distinct points. There must exist  $E' \in |K - a|$  such that  $t^{-1}(E') = t'^{-1}(E)$ . This means that for each decomposition  $E = d + d'$  with  $d, d'$  in  $\text{Sym}^2 C$ , we have  $E' = \langle K - d \rangle + \langle K - d' \rangle$ .

Let us write  $\langle K - p - q \rangle = p' + q'$  and  $\langle K - r - s \rangle = r' + s'$ , so that  $E' = p' + q' + r' + s'$ . We must have  $E' = \langle K - p - r \rangle + \langle K - q - s \rangle$ , so we can suppose  $\langle K - p - r \rangle = p' + r'$ . Then  $K - p - p' \sim q + q' \sim r + r'$ , which implies  $r' = q$ ,  $q' = r$ . But then we get  $K - p - q - r \sim p'$  and  $a \sim s - p'$ , which contradicts the hypothesis  $a \notin \Sigma_{JC}$ .  $\blacksquare$

We can now conclude:

**Proposition 2.4.** *For  $a \notin \Sigma_{JC}$ , the genus 4 curve  $X_a$  is not hyperelliptic; the unique quadric  $Q \subset \mathbb{P}^3$  containing its canonical model is smooth and split over  $k$  (that is, isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  over  $k$ ).*

*Proof :* Since  $X_a$  admits a base point free  $g_3^1$  it cannot be hyperelliptic ([ACGH], p. 13). Let us denote the two distinct  $g_3^1$  of  $X_a$  by  $|E|$  and  $|E'|$ . We have  $E + E' \sim K_{X_a}$ ; by the base-point free pencil trick, the multiplication map  $H^0(X_a, E) \otimes H^0(X_a, E') \rightarrow H^0(X_a, K_{X_a})$  is an isomorphism. Thus the canonical map of  $X_a$  is the composition of  $(t, t') : X_a \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  and of the Segre embedding  $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ , so the unique quadric containing the image is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ . ■

*Remark 2.5.* If  $C$  is hyperelliptic, Proposition 2.1 still holds, with essentially the same proof. However Lemma 2.3 fails: in fact, we have  $t' = \sigma \circ t$ , where  $\sigma : |K + a| \xrightarrow{\sim} |K - a|$  is induced by the hyperelliptic involution. Actually in that case  $X_a$  has a unique  $g_3^1$ , at least if  $\text{char}(k) \neq 2$ . Indeed  $\Theta$  has a singular point, given by the  $g_2^1$  of  $C$ ; on the other hand  $JC$  is isomorphic to the Prym variety of  $\tilde{X}_a/X_a$ . By [M2], §7, Thm. (c), this happens if and only if  $X_a$  admits a unique  $g_3^1$ .

2.6. The divisor  $\Sigma'_{JC}$  is equal to  $\mathbf{2}_* \Delta$ , where  $\mathbf{2}$  is the endomorphism  $z \mapsto 2z$  of  $JC$  and  $\Delta$  is any symmetric theta divisor; thus it can be defined on any absolutely indecomposable principally polarized abelian threefold  $(A, \theta)$ , with no reference to the isomorphism  $A \xrightarrow{\sim} JC$ . The same holds for  $\Sigma''_{JC}$  provided  $\text{char}(k) \neq 2$ . Recall indeed that there is a canonical linear system on  $A$ , denoted  $|2\theta|$ , which contains the double of each symmetric theta divisor. Then:

**Lemma 2.7.** *If  $\text{char}(k) \neq 2$ , the divisor  $\Sigma''_{JC} = C - C$  is the unique divisor in  $|2\theta|$  with multiplicity  $\geq 4$  at 0.*

This is quite classical if  $k = \mathbb{C}$ , see [GG]. We do not know whether it still holds when  $\text{char}(k) = 2$ .

*Proof :* The difference map  $C \times C \rightarrow C - C$  is an isomorphism outside the diagonal  $\Delta$ , and contracts  $\Delta$  to 0; therefore the multiplicity of  $C - C$  at 0 is  $-\Delta^2 = 4$ .

Let us prove the unicity; we may assume  $k = \bar{k}$ . We denote by  $|2\theta|_0$  the subspace of elements of  $|2\theta|$  containing 0. The multiplicity at 0 of an element of  $|2\theta|$  is even: this follows from the “inverse formula” of [M1], p. 331. Thus we have a projective linear map  $\tau : |2\theta|_0 \dashrightarrow |\mathcal{O}_{\mathbb{P}^2}(2)|$  which associates to a divisor its quadratic tangent cone at 0. Since  $\dim |2\theta|_0 = 6$

and  $\dim |\mathcal{O}_{\mathbb{P}^2}(2)| = 5$ , it suffices to prove that  $\tau$  is surjective. For each  $E \in \text{Sym}^2 C$ , the divisor  $(\text{Sym}^2 C - E) + (\text{Sym}^2 C - (K - E))$  belongs to  $|2\theta|_0$ ; by [K], thm. 2, its tangent cone at 0 is twice the line in  $\mathbb{P}^2$  spanned by  $E$ . Since the double lines span the space of conics,  $\tau$  is surjective. ■

### 3. THE MAIN RESULT

In this section we fix a principally polarized abelian threefold  $(A, \theta)$  over  $k$ . We assume that it is absolutely indecomposable, that is,  $(A, \theta)$  is not isomorphic over  $\bar{k}$  to a product of two principally polarized abelian varieties. It is equivalent to say that the theta divisor of  $A$  is irreducible (over  $\bar{k}$ ), or that  $(A, \theta)_{\bar{k}}$  is isomorphic to the Jacobian of a curve [OU]. This does not imply that  $(A, \theta)$  itself is a Jacobian; indeed we have [S]:

**Proposition 3.1.** *There exists a curve  $C$  over  $k$  and a character  $\varepsilon_A : \text{Gal}(\bar{k}/k) \rightarrow \{\pm 1\}$ , uniquely determined, such that  $(A, \theta)$  is  $k$ -isomorphic to  $JC$  twisted by  $\varepsilon_A$ . If  $C$  is hyperelliptic,  $\varepsilon_A$  is trivial.*

3.2. In more down-to-earth terms this means the following. Let  $k'$  be the extension of  $k$  defined by the character  $\varepsilon_A$ . Then:

- if  $\varepsilon_A = 1$  (that is,  $k' = k$ ),  $JC$  is  $k$ -isomorphic to  $(A, \theta)$ . This is the case if  $C$  is hyperelliptic.
- if  $\varepsilon_A \neq 1$  (that is,  $k'$  is a quadratic extension of  $k$ ),  $JC$  is isomorphic to  $(A, \theta)$  over  $k'$  but not over  $k$ . More precisely, let  $\sigma$  be the nontrivial automorphism of  $k'/k$ ; there exists an isomorphism  $\varphi : (A, \theta) \rightarrow JC$  such that  $\sigma\varphi = -\varphi$ .

3.3. Our aim is to describe geometrically the character  $\varepsilon_A$ . We will compare it to the character associated to a smooth quadric  $Q \subset \mathbb{P}_k^3$  in the following way: such a quadric admits two rulings defined over  $\bar{k}$ , so the action of  $\text{Gal}(\bar{k}/k)$  on these rulings provides a character  $\varepsilon_Q : \text{Gal}(\bar{k}/k) \rightarrow \{\pm 1\}$ . We will describe this character in more concrete terms below.

We define the divisor  $\Sigma_A = \Sigma'_A \cup \Sigma''_A$  on  $A$  as in 2.6: we put  $\Sigma'_A = 2_*\Delta$  for any symmetric theta divisor  $\Delta$ ; if  $\text{char}(k) \neq 2$ ,  $\Sigma''_A$  is the unique divisor in  $|2\theta|$  with multiplicity  $\geq 4$  at 0. An alternative definition, which works in all characteristics, is as follows: we choose an isomorphism  $A \xrightarrow{\sim} JC$  over  $k'$  and put  $\Sigma''_A = \varphi^{-1}(C - C)$ . Since  $C - C$  is symmetric this definition does not depend on the choice of  $\varphi$ .

We assume that  $A$  admits a theta divisor  $\Theta$  defined over  $k$ . If  $\Theta$  is singular,  $C$  is hyperelliptic, hence  $A \cong JC$  by Proposition 3.1. Thus we may assume that  $\Theta$  is smooth.

We also assume that there exists a point  $a \in A(k)$  outside  $\Sigma_A$ . The divisor  $-\Theta$  is in the class of the polarization  $\theta$ , hence there is a unique  $b \in A(k)$  such that  $(-\Theta) + b = \Theta + a$ ; the involution  $z \mapsto b - z$  exchanges  $\Theta$  and  $\Theta + a$ .

**Theorem 3.4.** *Let  $X_a$  be the quotient of the curve  $\Theta \cap (\Theta + a)$  by the involution  $z \mapsto b - z$ . Then  $X_a$  is a smooth curve of genus 4, non hyperelliptic. Its canonical model lies in a smooth quadric  $Q \subset \mathbb{P}^3$ , and we have  $\varepsilon_A = \varepsilon_Q$ .*

*Proof :* Following 3.2 we choose an isomorphism  $\varphi : (A, \theta) \rightarrow JC$  defined over  $k'$ . It induces an isomorphism of  $\tilde{X}_a$  onto the corresponding curve  $\tilde{X}_{\varphi(a)} \subset JC$ , hence of  $X_a$  onto  $X_{\varphi(a)}$ . By remark 2.6  $\varphi$  maps  $\Sigma_A$  onto  $\Sigma_{JC}$ , thus  $\varphi(a) \notin \Sigma_{JC}$ ; then Proposition 2.4 tells us that  $X_a$  is not hyperelliptic and that its canonical model is contained in a smooth quadric  $Q \subset \mathbb{P}^3$  which is split over  $k'$ . This means that the character  $\varepsilon_Q$  is trivial on the subgroup  $\text{Gal}(\bar{k}/k')$  of  $\text{Gal}(\bar{k}/k)$ ; in other words,  $\varepsilon_Q$  is either trivial or equal to  $\varepsilon_A$ .

It remains to prove that  $\varepsilon_Q$  is nontrivial when  $k' \neq k$ , that is, the nontrivial automorphism  $\sigma$  of  $k'/k$  exchanges the two rulings of  $Q$ , or equivalently the two  $g_3^1$  of  $X_a$ .

We have  ${}^\sigma\varphi = -\varphi$  (3.2). We write as before  $\varphi(\Theta) = \text{Sym}^2 C - D$ ; we observe that  ${}^\sigma(\varphi(\Theta)) = -\varphi(\Theta)$ , hence  ${}^\sigma D \sim K - D$ . Recall that the maps  $t : X_a \rightarrow |K + \varphi(a)|$  and  $t' : X_a \rightarrow |K - \varphi(a)|$  defining the two  $g_3^1$  are given by

$$\begin{aligned} t(\bar{z}) &= \langle D + \varphi(z) \rangle + \langle K - D - \varphi(z) + \varphi(a) \rangle \\ t'(\bar{z}) &= \langle K - D - \varphi(z) \rangle + \langle D + \varphi(z) - \varphi(a) \rangle, \end{aligned}$$

where  $z$  is a point of  $\tilde{X}_a$  and  $\bar{z}$  its image in  $X_a$ .

Using  ${}^\sigma\varphi = -\varphi$  and  ${}^\sigma D \sim K - D$  we get

$${}^\sigma t(\bar{z}) = \langle K - D - \varphi(z) \rangle + \langle D + \varphi(z) - \varphi(a) \rangle = t'(\bar{z});$$

thus  $\sigma$  exchanges  $t$  and  $t'$ , hence the two rulings of  $Q$ . ■

One can describe the extension  $k'/k$  (hence the character  $\varepsilon_Q$ ) using the even Clifford algebra  $C^+(Q)$  [De]: its center is isomorphic to  $k'$  if  $k' \neq k$  and to  $k \times k$  otherwise. From the description of this center (see [Bo], §9, no. 4, Remarque 2), we obtain:

**Proposition 3.5.** *Assume  $\text{char}(k) \neq 2$ , and let  $\delta \in k^*$  be the discriminant of  $Q$  (well defined mod.  $k^{*2}$ ). The extension  $k'$  is isomorphic to  $k(\sqrt{\delta})$ .*

Similarly, if  $\text{char}(k) = 2$ , we have  $k' = k(\lambda)$  with  $\lambda^2 + \lambda = \Delta$ , where  $\Delta$  is the pseudo-discriminant of  $Q$  ([Bo], §9, exerc. 9).

Finally let us observe that the existence of a rational theta divisor is automatic when  $k$  is finite. Indeed the theta divisors in the class  $\theta$  form a torsor under  $A$ ; by a theorem of Lang [L], such a torsor is trivial.

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